An Efficient Lyapunov Equation-Based Approach for Generating Reduced-Order Models of Interconnect

Jing-Rebecca Li Frank Wang Jacob White
Research Laboratory of Electronics
Massachusetts Institute of Technology
Cambridge, MA 02139
Email: {jingli, frank, white}@rle-vlsi.mit.edu

Abstract

In this paper we present a new algorithm for computing reduced-order models of interconnect which utilizes the dominant controllable subspace of the system. The dominant controllable modes are computed via a new iterative Lyapunov equation solver, Vector ADI. This new algorithm is as inexpensive as Krylov subspace-based moment matching methods, and often produces a better approximation over a wide frequency range. A spiral inductor and a transmission line example show this new method can be much more accurate than moment matching via Arnoldi.

1 Introduction

Designers of analog and high performance digital integrated circuits rely heavily on circuit-level simulation programs which can efficiently incorporate accurate models of the interconnect. The now-standard approach to efficient circuit-interconnect simulation is to represent the interconnect with moment matching-based reduced-order models [19, 2, 9]. Accurate computation of such models can be accomplished using bi-orthogonalization algorithms like Padé via Lanczos (PVL) [5], or with methods based on orthogonalized Krylov subspace methods [17, 1, 20].

Another approach to computing these reduced-order models is the Truncated Balanced Realization (TBR) [8]. TBR produces a reduced model which is often close to the optimal Hankel-norm approximation, and also has a known Ł$^\infty$-transfer function error bound. However, TBR has been largely abandoned for the interconnect model order reduction application, because it requires solving two Lyapunov equations for the controllability and observability grammians, as well as balancing the grammians using several matrix factorizations and products. These requirements made TBR too computationally expensive to use on large problems.

In this paper we propose reducing the system so only the dominant controllable subspace remains. This only requires the dominant singular subspace of the controllability grammian. We present a new algorithm, Vector ADI, to compute this dominant singular subspace. Vector ADI comes from reformulating the well-known ADI method for the case of Lyapunov equations with low rank right hand side. It is shown that Vector ADI in fact generates a particular rational Krylov subspace of the system matrix A and the input coefficient matrix B. This new method requires only linear matrix-vector solves, and hence enables one to take advantage of any system sparsity.

Section 2 gives brief background on state-space description of linear time-invariant systems, and system grammians as solution of low rank right hand side (LRRHS) Lyapunov equations. Section 3 describes two existing approaches to model order reduction, moment matching via Lanczos or Arnoldi and Truncated Balanced Realization. Section 4 describes reducing the system via the dominant controllable subspace. In section 5, we derive the Vector ADI method for computing the dominant controllable subspace, and show that it is equivalent to generating a rational Krylov subspace. Section 6 compares the work required for Vector ADI and for moment matching via Arnoldi (MMVA). In section 7 the new Vector ADI-controllable subspace method is compared with MMVA and TBR in two numerical examples. Section 8 contains concluding remarks.

2 State-Space Representation

A linear time-invariant system with realization (A, B, C) is characterized by the equations:

$$\dot{x} = Ax + Bu \qquad (1)$$
$$y = Cx \qquad (2)$$

where $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{p\times 1}$, and $y \in \mathbb{R}^{q\times 1}$ are the vectors of state variables, input, and output, respectively. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, are the system matrix, the input coefficient matrix, and the output coefficient matrix, respectively.

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In single-input single-output (SISO) systems, $p = 1, q = 1$. Even in multiple-input, multiple-output (MIMO) systems, p and q are both very small compared to the number of state variables n.

If the system matrix A is stable, ie, the eigenvalues of A are in the left open half plane, we can define the controllability grammian as

$$P \triangleq \int_0^\infty e^{At} BB^T e^{A^T t} dt \tag{3}$$

and the observability grammian as

$$Q \triangleq \int_0^\infty e^{A^T t} C^T C e^{At} dt. \tag{4}$$

It can be seen that the grammians P and Q are symmetric, and satisfy the following Lyapunov equations

$$AP + PA^T + BB^T = 0 \tag{5}$$
$$A^T Q + QA + C^T C = 0 \tag{6}$$

Since $rank(BB^T) = rank(B) \leq p \ll n$ and $rank(C^T C) = rank(C) \leq q \ll n$, equations (5-6) both have a low rank right hand side (LRRHS).

The grammians provide information about the reachability and observability of the system, and are needed in optimal Hankel-norm or Truncated Balanced Realization-type model reductions[4, 8, 18]. Frequently the bottleneck of these 'optimal' model reduction methods occurs in the solution of the two Lyapunov equations for the system grammians. So far, all methods of solving Lyapunov equations, exact or iterative, have been $O(n^3)$ work [3, 10, 15, 22].

3 Model Order Reduction

The system described by equations (1-2) is characterized by its transfer function $G(s)$,

$$G(s) = C(sI - A)^{-1} B, \quad Y(s) = G(s)U(s). \tag{7}$$

Model order reduction seeks to obtain a smaller system

$$\dot{x}_r = A_r x_r + B_r u \tag{8}$$
$$y_r = C_r x_r \tag{9}$$

such that the number of state variables of this new systems is much smaller than n, and the transfer function of the new system, $G_r(s)$,

$$G_r(s) = C_r(sI - A_r)^{-1} B_r, \quad Y_r(s) = G_r(s)U_r(s) \tag{10}$$

is close to the original.

3.1 Moment Matching Methods

Up to now, model order reduction of linear systems has usually gone in one of two directions. One is moment matching, which includes Padé, partial realization, and their shifted versions [7, 9, 11]. These methods usually utilize the Arnoldi or Lanczos method to find an orthonormal basis for some combination of Krylov subspaces, $K_J(A,B)$, $K_J(A^T,C^T)$, $K_J((A - pI)^{-1},B)$, or $K_J((A^T - pI)^{-1},C^T)$, where

$$K_J(A,B) = span\{B, AB, A^2B, \cdots, A^{(J-1)}B\}. \tag{11}$$

The result is that moments, when $K_J((A - pI)^{-1},B)$ is used, or Markov parameters, when $K_J(A,B)$ is used, of the reduced system match those of the original to a certain order.

The advantage of these methods is that they only require matrix-vector products or linear matrix solves, and hence are very efficient. When $(A - pI)^{-1}B$ is required, an iterative solver such as GMRES is often employed so still only matrix-vector products by A are needed [13]. However, there is no theoretical error bound for the reduced system's transfer function. The error will be small at points where moments or Markov parameters are matched, but there is no guarantee that the error will also be small elsewhere. It's more of an art than science to pick points where moments are to be matched so that the overall transfer function error is small. The algorithms for picking matching frequency points are based on heuristics [2]. Moreover, the reduced system obtained by these methods is not guaranteed to be passive or stable, even if the original system is. Further processing is needed to obtain a passive and stable model [17].

3.2 Truncated Balanced Realization

The other direction in model order reduction is Truncated Balanced Realization, which produces a guaranteed stable reduced system and has a theoretical transfer function error bound. The following summarizes the development in [8].

Given a stable system described by equations (1-2), with controllability and observability grammians, P and Q, respectively. Let Q have a Cholesky factorization

$$Q = R^T R \tag{12}$$

then RPR^T will be a positive-definite matrix and can be diagonalized as

$$RPR^T = U\Sigma^2 U^T, \qquad U^T U = I \tag{13}$$
$$\Sigma = diag(\sigma_1, \sigma_2, ..., \sigma_n), \qquad \sigma_1 \geq \sigma_2 ... \geq \sigma_n > 0 \tag{14}$$

A balancing transformation is given by

$$T = \Sigma^{-1/2} U^T R. \tag{15}$$

In the transformed state space coordinates, with realization $(A_b = TAT^{-1}, B_b = TB, C_b = CT^{-1})$, the new controllability and observability grammians are diagonal and equal,

$$P_b = Q_b = \Sigma = diag\{\sigma_1, \sigma_2, ..., \sigma_k, \sigma_{k+1}, ..., \sigma_n\} \tag{16}$$

If $\sigma_k > \sigma_{k+1}$, then the kth order truncated balanced realization is given by

$$(A^k_{tbr}, B^k_{tbr}, C^k_{tbr}) = (A_{11}, B_1, C_1) \tag{17}$$

where $A_{11} \in \mathbb{R}^{k \times k}, B_1 \in \mathbb{R}^{k \times p}, C_1 \in \mathbb{R}^{q \times k}$ are sub-matrices of the balanced realization, (A_b, B_b, C_b),

$$A_b = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B_b = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C_b = \begin{pmatrix} C_1 & C_2 \end{pmatrix}. \tag{18}$$

The resulting transfer function $G^k_{tbr}(s)$ has L^∞-error

$$\|G(jw) - G^k_{tbr}(jw)\|_{L^\infty} \leq 2(\sigma_{k+1} + \sigma_{k+2} + ... + \sigma_n). \tag{19}$$

4 Reduction via the Dominant Controllable Subspace

Solving the Lyapunov equations (5-6) is expensive, as is balancing the grammians. Instead, we propose reducing the system to its dominant controllable subspace only, and ignore observability. Both grammians will be taken into account in a later paper [14].

Suppose $P = U_p \Sigma_p U_p^T$ is the singular value decomposition of the controllability grammian. Under the coordinate transformation, $\tilde{x} \triangleq U_p^T x$, the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

becomes

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}u && (20) \\ y &= \tilde{C}\tilde{x} && (21) \\ \tilde{A} &= U_p^T A U_p, \tilde{B} = U_p^T B, \tilde{C} = CU_p. && (22) \end{aligned}$$

The controllability and observability grammians become $\tilde{P} = U_p^T P U_p = \Sigma_p$ and $\tilde{Q} = U_p^T Q U_p$. Clearly the system transfer function is not affected by this invertible coordinate transformation.

Suppose the diagonal of Σ_p is in decreasing order, we can then truncated the new realization $(\tilde{A}, \tilde{B}, \tilde{C})$ to the first k coordinates, which are precisely the k most controllable modes. Partition U_p as $[U_p^k, U_p^{n-k}]$, U_p^k being the k most controllable modes of the original realization (A, B, C), the reduced system is

$$\begin{aligned} \dot{x}_r^k &= A_r^k x_r^k + B_r^k u && (23) \\ y_r &= C_r^k x_r^k && (24) \\ A_r^k &= {U_p^k}^T A U_p^k, B_r^k = {U_p^k}^T B, C_r^k = CU_p^k. && (25) \end{aligned}$$

Therefore, to obtain a kth order reduced system, only the k-dim dominant controllable subspace U_p^k is needed. Note that (25) is in the form of a congruence transformation, which has a side benefit of aiding in the preservation of passivity [12]. In the next section we describe how U_p^k can be obtained efficiently via Vector ADI.

5 Vector ADI

We seek a good approximation to the dominant singular subspace U_p^k of the solution P of equation (5). We start with the original ADI method.

5.1 The ADI Method

The Alternate Direction Implicit method is an iterative method for solving the Lyapunov equation,

$$AX + XA^T + D = 0. \quad (26)$$

Formulated in [3, 15], it has the form:

$$\begin{aligned} X_0 &= 0; && (27) \\ (p_j I - A) X_{j-\frac{1}{2}} &= D + X_{j-1}(A^T + p_j I), && (28) \\ (p_j I - A) X_j &= D + X_{j-\frac{1}{2}}^T (A^T + p_j I). && (29) \end{aligned}$$

A must first be reduced to tridiagonal form in $O(n^3)$ work for ADI to be competitive with standard techniques. The flop count for ADI calculated in [15] is $\frac{19}{3}n^3 + 12Jn^2$ where J is the number of ADI iterations.

If X_J is the ADI approximation after J iterations, the error is bounded by,

$$\begin{aligned} &\|X_J - X\|_F \le \|T\|_2^2 \|T^{-1}\|_2^2 k(\mathbf{p})^2 \|X_0 - X\|_F, \\ &k(\mathbf{p}) = \max_{x \in spec(A)} \left| \prod_{j=1}^{J} \frac{(p_j - x)}{(p_j + x)} \right|, \end{aligned} \quad (30)$$

where T is a matrix of eigenvectors of A, and $\mathbf{p} = \{p_1, p_2, \cdots, p_J\}$ are the ADI parameters. Parameter selection for ADI was investigated in several papers, [3, 15, 21, 22]. It was noted in [15] that with good parameters ADI usually converges in a few iterations.

5.2 Vector ADI Derivation

The key in developing Vector ADI lies in an alternate formulation of the ADI method [3].

$$X_0 = 0; \quad (31)$$

$$\begin{aligned} X_j = &2p_j(A - p_j I)^{-1} D (A - p_j I)^{-T} \\ &+ (A - p_j I)^{-1}(A + p_j I) X_{j-1} (A + p_j I)^T (A - p_j)^{-T}. \end{aligned} \quad (32)$$

Since $D = BB^T$ for equation (5), it is clear from (32) that X_j will always be symmetric, and that it will have rank at most the sum of the ranks of X_{j-1} and B. Since iteration starts with $X_0 = 0$, X_j will have rank at most jp, p being the number of vectors in B. Therefore, X_j needs not be represented by more than jp vectors.

We write $X_j = V_j {V_j}^T$, where $V_j = [v_1, v_2, \cdots, v_j]$ is a matrix square root of X_j, and each $v_i \in \mathbb{R}^{n \times p}$ is the same size as B. Replacing X_j by $V_j {V_j}^T$, and D by BB^T in equations (31) and (32), we obtain,

$$V_0 = 0; \quad (33)$$

$$\begin{aligned} V_j {V_j}^T = &2p_j(A - p_j I)^{-1} BB^T (A - p_j I)^{-T} \\ &+ (A - p_j I)^{-1}(A + p_j I) V_{j-1} V_{j-1}^T (A + p_j I)^T (A - p_j)^{-T}. \end{aligned} \quad (34)$$

It becomes clear that the algorithm can be reformulated in terms of the matrix square root V_j. There is no need to calculate X_j at each iteration, only V_j, which can be chosen so that it is easily found from V_{j-1}.

Here is the preliminary form of Vector ADI:

$$V_1 = \sqrt{2p_1}(A - p_1)^{-1} B, \quad V_1 \in \mathbb{R}^{n \times p} \quad (35)$$

$$\begin{aligned} &V_j = [\sqrt{2p_j}(A - p_j I)^{-1} B, (A - p_j I)^{-1}(A + p_j I) V_{j-1}] \\ &V_j \in \mathbb{R}^{n \times jp} \end{aligned} \quad (36)$$

In this form, at each step, the number of vectors needing to be modified is increased by p. The next step in developing the algorithm involves keeping the number of vectors modified at each step constant.

5.3 Rational Krylov Subspace Formulation

In the original ADI method, the number of iterations needed to achieve a required error tolerance is determined a priori [15]. Then the ADI parameters $\{p_j\}$ are calculated as a function of the required number of iterations and A's spectral bounds.

Suppose the number of iterations to be performed is J, it is then possible to write Vector ADI in a form which requires only p vectors to be modified at each iteration, p being the number of vectors in B. The Jp vectors of V_J are,

$$\begin{aligned} V_J = [& S_J\sqrt{2p_J}B, \quad S_JT_JS_{J-1}\sqrt{2p_{J-1}}B, \\ & S_JT_JS_{J-1}T_{J-1}S_{J-2}\sqrt{2p_{J-2}}B, \quad \cdots, \\ & S_JT_J\cdots S_2T_2S_1\sqrt{2p_1}B] \end{aligned} \tag{37}$$

$$S_i = (A - p_iI)^{-1}, \qquad T_i = (A + p_iI). \tag{38}$$

It's easily shown that S_i and T_k commute for any i and k. Define:

$$w_J = \sqrt{2p_J}S_JB = \sqrt{2p_J}(A - p_JI)^{-1}B \tag{39}$$

$$P_l = \frac{\sqrt{2p_l}}{\sqrt{2p_{l+1}}}S_lT_{l+1} \tag{40}$$

$$= \frac{\sqrt{2p_l}}{\sqrt{2p_{l+1}}}[I + (p_{l+1} + p_l)(A - p_lI)^{-1}] \tag{41}$$

V_J then becomes:

$$\begin{aligned} V_J = [& w_J, \quad P_{J-1}w_J, \quad P_{J-2}P_{J-1}w_J, \quad \cdots, \\ & P_1P_2\cdots P_{J-1}w_J] \end{aligned} \tag{42}$$

In this form, V_J can be obtained from the starting vector w_J, (which is actually a p-vector, the same size as B), and $J-1$ products of the form P_iw. The cost of applying P_i to a vector is essentially that of a single linear matrix-vector solve. The starting vector w_J is obtained from a linear matrix solve with B as the right-hand side, and each succeeding p-vector can be obtained from the previous p-vector at the cost of a linear matrix solve.

It can be seen that the columns of V_J span a rational Krylov subspace, $K(w_J, \mathbf{P}(A), J)$, with starting vector $w_J = \sqrt{2p_J}(A - p_JI)^{-1}B$ and successive matrix products by $P_i(A)$, which are non-identical rational functions of A.

The ADI solution is then $\tilde{X} = V_JV_J^T$. If $k \le J$, it can be easily seen that U^k, the k-dim dominant singular space of $\tilde{X}$, is the same as the k-dim dominant left singular space of V_J, $V_J = [U^k, U^{J-k}]\Sigma_{V_J}W_{J\times J}$, which can be obtained cheaply because V_J contains only Jp vectors.

If A is the system matrix and B the input coefficient matrix, then U^k is the desired approximate dominant controllable subspace in equations (23 -25).

6 Work Comparison with Moment Matching

If J iterations of Vector ADI are performed, the matrix square root V_J of the ADI solution can be obtained after J linear matrix solves and $J-1$ vector additions. Moment matching via Arnoldi [16, 11], calculates the following Krylov subspace,

$$\begin{aligned} & K_J((A - pI)^{-1}, B) \\ & = span\{B, (A - pI)^{-1}B, \cdots, (A - pI)^{-(J-1)}B\} \end{aligned} \tag{43}$$

which requires $J-1$ linear matrix solves. Therefore, Arnoldi and Vector ADI require comparable work to generate a given order model.

7 Numerical Examples

The Vector ADI-controllable subspace method was tested on several systems with large sparse state-space matrices, and compared with TBR and moment matching around $s = 0$ via Arnoldi (MMVA).

The first example comes from inductance extraction of an on-chip planar square spiral inductor suspended over a copper plane. This example was used in [11] to demonstrate a combined moment matching and TBR method. The spiral inductor was originally discretized into a 500×500 system and good approximation is expected by a reduced model of order 10 or so.

Reduced models of order 7 obtained by TBR, MMVA, and Vector ADI (in solid), are shown in Figure 1. It can be seen that even though MMVA around $s = 0$ gives very accurate frequency response near $s = 0$, it loses accuracy away from it. TBR seeks to minimize the maximum error in the frequency response, and hence shows similar error along the entire frequency range and looks flat. Vector ADI follows TBR throughout the entire range of frequencies and has the same flat shape. It gives somewhat larger error than TBR for both resistance and inductance, but is much better than MMVA.

The work required for each of the three methods is given in table 1.

	TBR	MMVA	VADI
Flop Count	10.2e9	0.60e9	0.67e9

Table 1: Spiral Inductor Flop Count

VADI and MMVA require about the same amount of work, which is around 6 percent of the work required for TBR.

The results for this example are quite encouraging and show that VADI can produce a reduced model that has uniformly small error over a wide frequency range at a small percentage of the cost of TBR.

The second example comes from the discretization of a transmission line using the formulation in [16], with the original system having 256 states. In Figure 2 Vector ADI clearly captures the behavior of both the amplitude and phase of the frequency response much better than MMVA. With a 20th order system, Vector ADI is able to capture 5 peaks of the amplitude, whereas MMVA only the first 2 and somewhat of the 4th. Vector ADI also matches more peaks in the phase of the original response, (with an extra peak at the end), whereas the phase from MMVA essentially lies flat after the first dip.

8 Conclusions and Acknowledgments

In this paper we presented a new algorithm for computing reduced-order models of interconnect, using the dominant controllable subspace of a system, which is computed via a new low rank right hand side Lyapunov equation solver, Vector ADI. This method of model reduction requires only shifted linear matrix solves, and hence enables one to take advantage of any system sparsity. We demonstrated that this Vector ADI-controllable subspace method produces much better models than the moment matching approach if wide-band fidelity is important.

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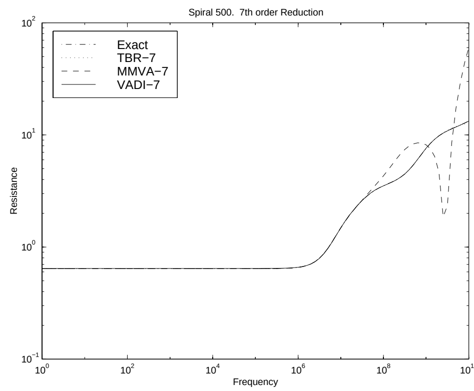
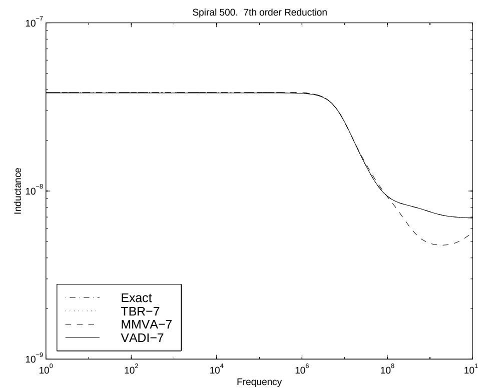
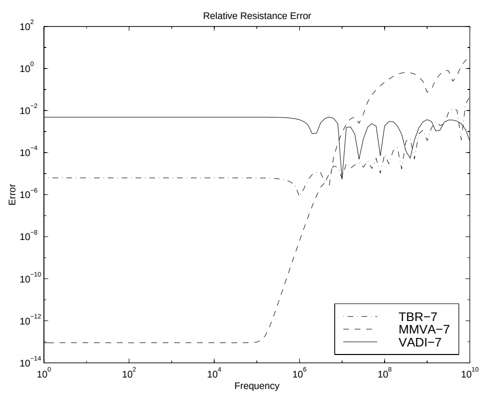


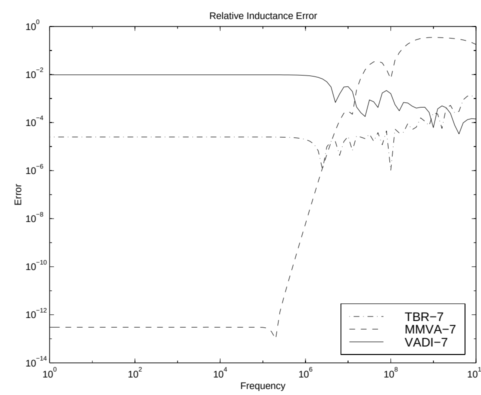
(a) Resistance



(b) Inductance

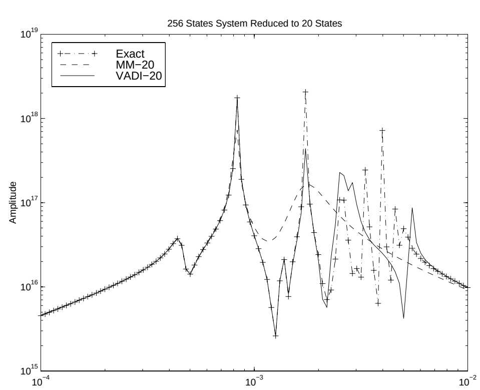


(c) Resistance Error

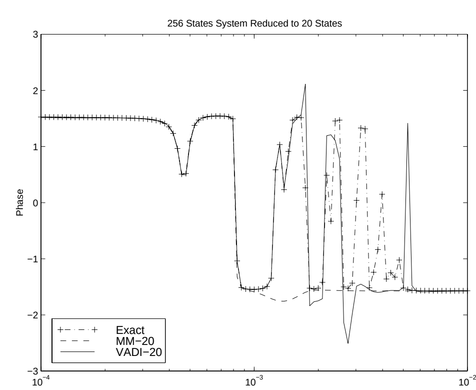


(d) Inductance Error

Figure 1: **Spiral Inductor. Vector ADI is seen to follow TBR throughout the frequency range.**



(a) Amplitude



(b) Phase

Figure 2: **Transmission Line. Vector ADI is seen to match significantly more peaks than moment matching around** $s = 0$**.**